5.(0) a) $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$

Calculo autovalores de A^TA:

$$P(\lambda) = \det\left(\begin{bmatrix} \lambda - 9 & 0 \\ 0 & \lambda \end{bmatrix}\right) = \lambda^2 - 9\lambda = \lambda \cdot (\lambda - 9)$$

$$\boxed{\text{Autoval: } \to \lambda_1 = 0 \quad \to \lambda_2 = 9}$$

Los valores singulares son:

$$\boxed{\sigma_1 = \sqrt{9} = 3 \qquad \sigma_2 = 0}$$

Calculo autovectores:

Para $\lambda = 9$

$$\begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix} \quad \{ y = 0 \to \bar{x} = x \cdot \underbrace{(1,0)}_{\substack{\text{AVECT.} \\ \lambda = 9}}$$

Para $\lambda = 0$

$$\begin{pmatrix} -9 & 0 \\ 0 & 0 \end{pmatrix} \quad \{ x = 0 \to \bar{x} = y \cdot \underbrace{(0,1)}_{\substack{\text{AVECT.} \\ \lambda = 0}}$$

Los normalizo y formo la matriz V:

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ahora calculo columnas de U: $v_1=(1,0)$ $v_2=(0,1)$

$$u_1=\frac{A.v_1}{\sigma_1}=\frac{\begin{bmatrix}-3&0\\0&0\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}}{3}=\begin{bmatrix}-1\\0\end{bmatrix}$$

~~Entonces~~ tomo U_2 ortonormal de $\mathbb{R}^2$ completando base: $U_2=\begin{bmatrix}0\\1\end{bmatrix}$

Luego:

$$\boxed{A=U\Sigma V^T=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}3&0\\0&0\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}}$$

b) $A=\begin{bmatrix}1&i\\i&-1\end{bmatrix}$ $A^*A=\begin{bmatrix}2&2i\\-2i&2\end{bmatrix}$

$$p(\lambda)=\det\left(\begin{bmatrix}\lambda-2&-2i\\2i&\lambda-2\end{bmatrix}\right)=\lambda^2-4\lambda+\cancel{4}-\underbrace{(-4i^2)}_{4}=\lambda^2-4\lambda=\lambda.(\lambda-4)$$

$$\boxed{\text{Autoval:}\ \lambda_1=0,\ \lambda_2=4}$$

$$\boxed{\sigma_1=\sqrt{4}=2\quad\sigma_2=0}$$

Para $\lambda=4$

$$\begin{pmatrix}2&-2i\\2i&2\end{pmatrix}\xrightarrow{F_2\to F_1.i-F_2}\begin{pmatrix}2&-2i\\0&0\end{pmatrix}\quad 2x-2iy=0\to x=iy$$

$$\to\bar{x}=y.\underbrace{(i,1)}_{\substack{\text{AVECT.}\\ \lambda=4}}$$

Para $\lambda=0$

$$\begin{pmatrix}-2&-2i\\2i&-2\end{pmatrix}\xrightarrow{F_2\to F_1.i+F_2}\begin{pmatrix}-2&-2i\\0&0\end{pmatrix}\quad -2x-2iy=0\to x=-iy$$

$$\to\bar{x}=y.\underbrace{(-i,1)}_{\substack{\text{AVECT.}\\ \lambda=0}}$$

Los normalizo y armo V:

$$V=\begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad v_1=\left(\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad v_2=\left(\frac{-i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Calculo U:

$$U_1 = \frac{A.v_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}\cdot\begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{2} = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

~~[illegible]~~ Tomo $U_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}$ para formar base ortonormal ~~de~~

Luego:

$$\boxed{A = U\Sigma V^{*} = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}}$$

c) $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$$P(\lambda) = \det \left(\begin{bmatrix} \lambda-8 & -2 \\ -2 & \lambda-5 \end{bmatrix} \right) = \lambda^2 - 13\lambda + 36$$

$$\boxed{\text{Autoval:} \to \lambda_1 = 9,\ \to \lambda_2 = 4}$$

$$\boxed{\sigma_1 = \sqrt{9} = 3 \qquad \sigma_2 = \sqrt{4} = 2}$$

Para $\lambda = 9$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} F2 \to 2F1 + F2 \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad x - 2y = 0 \to x = 2y$$

$$\to \bar{x} = y \cdot \underbrace{(2,1)}_{\text{AVECT. } \lambda = 9.}$$

Para $\lambda = 4$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} F2 \to F1 - 2F2 \begin{pmatrix} -4 & -2 \\ 0 & 0 \end{pmatrix} \quad -4x - 2y = 0 \to 4x = -2y \to y = -2x$$

$$\to \bar{X} = x \cdot \underbrace{(1,-2)}_{\text{AVECT. } \lambda = 4.}$$

Los normalizo y armo V:

$$V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \qquad v_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \qquad v_2 = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right)$$

Armo U:

$$U_1 = \frac{A \cdot v_1}{\sigma_1} = \frac{\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}{3} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$U_2 = \frac{A \cdot v_2}{\sigma_2} = \frac{\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}}{2} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}$$

forman BON de $\mathbb{R}^2$.

Luego:

$$\boxed{A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}}$$

10) d) $A = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$, $A^T A = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$

$P(\lambda) = \det\left(\begin{bmatrix} \lambda-74 & -32 \\ -32 & \lambda-26 \end{bmatrix}\right) = \lambda^2 - 100\lambda + 900$.

$$\boxed{\text{Autovalores: } \lambda_1 = 90,\ \lambda_2 = 10}$$

$$\boxed{\sigma_1 = \sqrt{90} \qquad \sigma_2 = \sqrt{10}}$$

Para $\lambda = 90$

$\begin{pmatrix} 16 & -32 \\ -32 & 64 \end{pmatrix} F_2 \to 2F_1 + F_2 \begin{pmatrix} 16 & -32 \\ 0 & 0 \end{pmatrix}$ $16x = 32y$ $\to x = 2y$

$\to \bar{x} = y . \boxed{\underbrace{(2,1)}_{\substack{\text{AVECT.} \\ \lambda=90}}}$

Para $\lambda = 10$

$\begin{pmatrix} -64 & -32 \\ -32 & -16 \end{pmatrix} F_2 \to F_1 - 2F_2 \begin{pmatrix} -64 & -32 \\ 0 & 0 \end{pmatrix}$ $-64x = 32y \to y = -2x$

$\to \bar{x} = x . \boxed{\underbrace{(1,-2)}_{\substack{\text{AVECT.} \\ \lambda=10}}}$

Los normalizo y armo V:

$$V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

$v_1 \quad v_2$

Armo U:

$$U_1 = \frac{\begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}{\sqrt{90}} = \frac{\begin{bmatrix} 15/\sqrt{5} \\ 0 \\ 15/\sqrt{5} \end{bmatrix}}{\sqrt{90}} = \begin{bmatrix} 15/\sqrt{450} \\ 0 \\ 15/\sqrt{450} \end{bmatrix}$$

$$U_2 = \frac{\begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}}{\sqrt{10}} = \frac{\begin{bmatrix} 5/\sqrt{5} \\ 0 \\ -5/\sqrt{5} \end{bmatrix}}{\sqrt{10}} = \begin{bmatrix} 5/\sqrt{50} \\ 0 \\ -5/\sqrt{50} \end{bmatrix}$$

Busco un vector ortogonal a los dos con prod. vectorial:

$$\begin{vmatrix} 15 & 0 & 15 \\ 5 & 0 & -5 \end{vmatrix} = (0, 150, 0)$$

Normalizado queda $U_3 = \begin{bmatrix} 0 \\ \frac{150}{\sqrt{22500}} \\ 0 \end{bmatrix}$

Luego $A = U \Sigma V^* \to$

$$\boxed{A = \begin{bmatrix} 15/\sqrt{450} & 5/\sqrt{50} & 0 \\ 0 & 0 & \frac{150}{\sqrt{22500}} \\ 15/\sqrt{450} & -5/\sqrt{50} & 0 \end{bmatrix} \cdot \begin{bmatrix} 90 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}}$$